Integrable Particle Dynamics in Accelerators

Tuesday: Integrable systems with static electromagnetic fields

Sergei Nagaitsev

Jan 29, 2019

Fermilab

Examples of Integrable systems

- Electric fields
  - Coulomb potential
  - Two Fixed Coulomb Centers
  - Vinti potential
- Magnetic fields
  - Uniform magnetic field
  - Magnetic monopole
- Electric and magnetic fields
  - Penning trap and its modifications

Kepler problem – a nonlinear integrable system

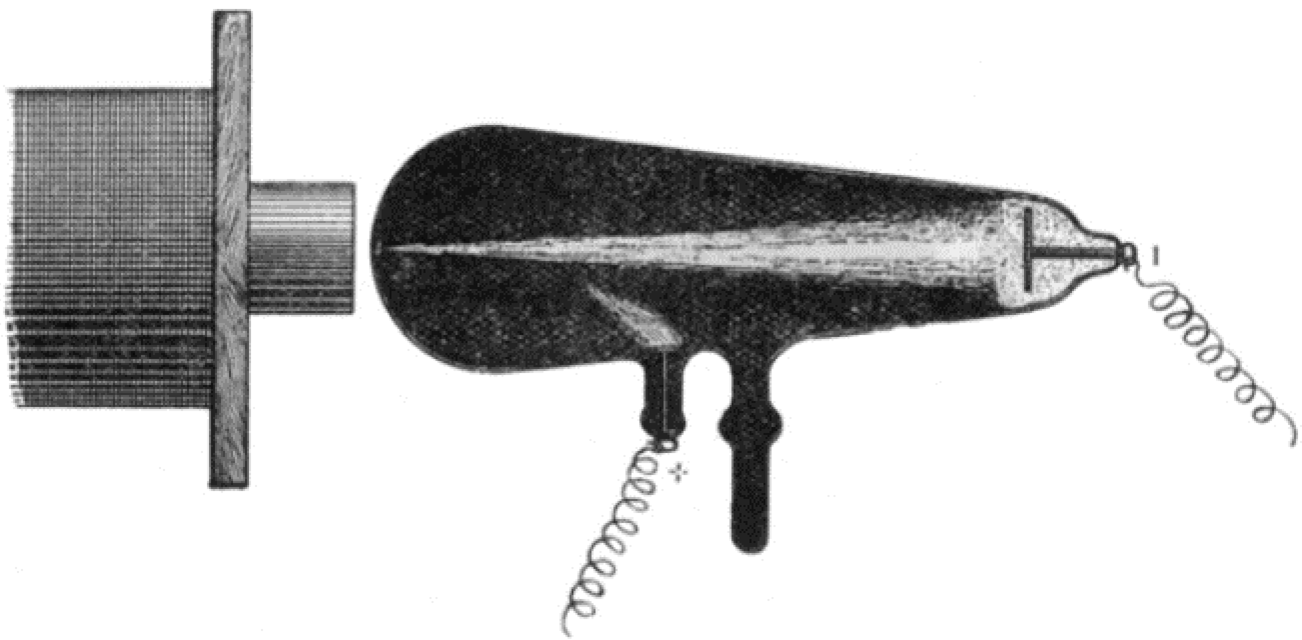
- Kepler problem (Coulomb potential): $$V = -\frac{k}{r}$$
- Separable in spherical coordinates:
  - For bounded motion

$$H = -\frac{mk^2}{2\left(J_r + J_\theta + J_\phi\right)^2}$$

- "Nonlinear" means that $H \neq \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3$
- Example of this system: the Solar system

Electric Charge in the Field of a Magnetic Pole

- Magnetic pole – "end" of a semi-infinite solenoid
- In 1896, Birkeland reported studies of cathode rays in a Crookes tube when a strong, straight electromagnet was placed outside and to the left.



- The nature of cathode rays was not yet understood

Kristian Birkeland

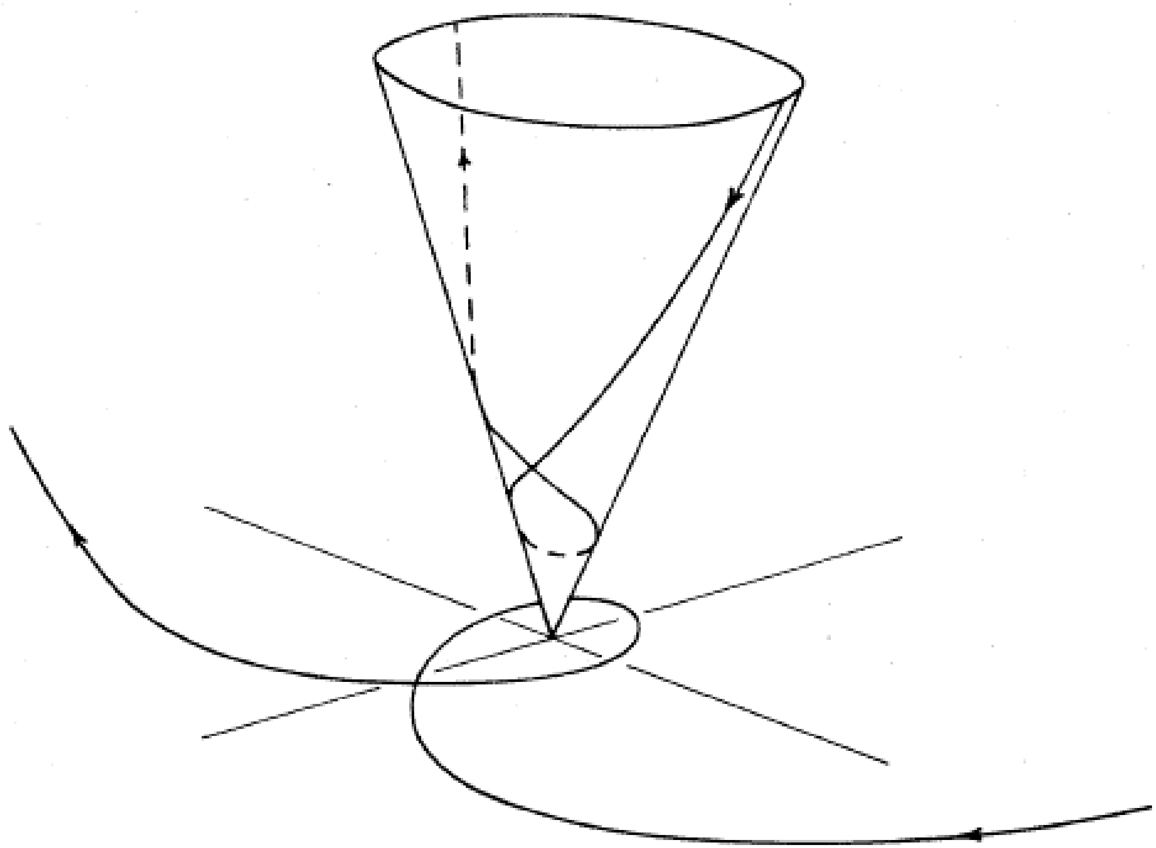
- Birkeland's scientific efforts are honored on the 200-kroner Norwegian banknote.
  - In 1896 his major interest was Aurora Borealis.
- He was one of Poincare's students in 1892

Magnetic monopole

- The nature of cathode rays was not understood in 1896, which were "discovered" to be electrons by J.J. Thomson in 1897 (in experiments with Crookes tubes and magnets).
- In 1896, before the Thomson's discovery, Poincare has suggested that Birkeland's experiment can be explained by "cathode rays being charges moving in the field of a magnetic monopole"
  - He wrote a brilliant paper in (1896), proving that charge motion in the field of magnetic monopole is fully integrable (but unbounded).

$$\mathbf{B} = \frac{k\mathbf{r}}{r^3}$$

Motion is on the cone surface



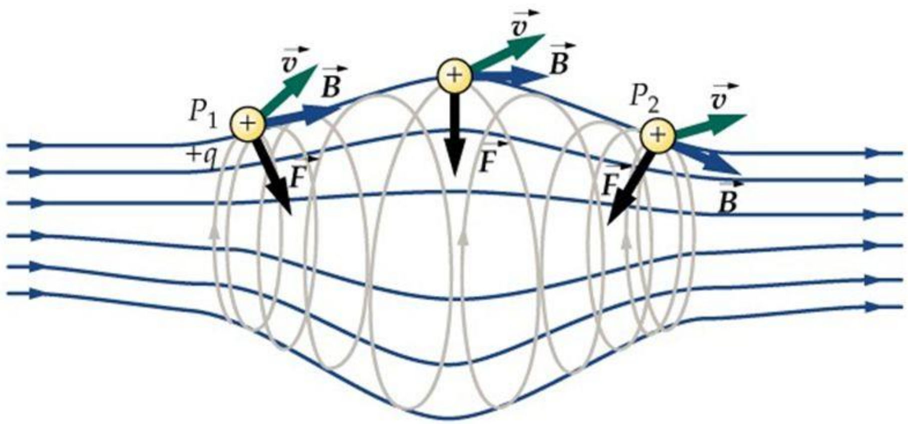
http://physics.princeton.edu/~mcdonald/examples/birkeland.pdf

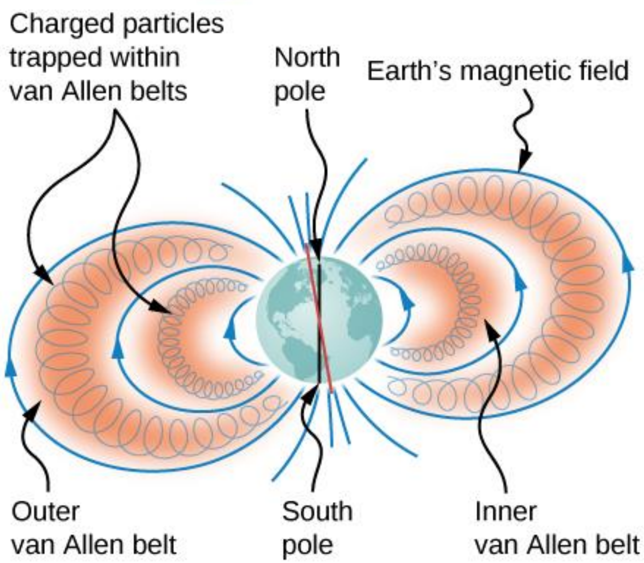
Aurora Borealis

- Beginning in 1904, a younger colleague, C. Størmer, was inspired by Birkeland to make extensive modeling of the trajectories of electrons in the Earth's magnetic field, approximated as that of a magnetic "bottle".
  - Størmer studied under Darboux and Poincare in 1898-1900

Størmer and Birkeland in 1910

Magnetic Bottle





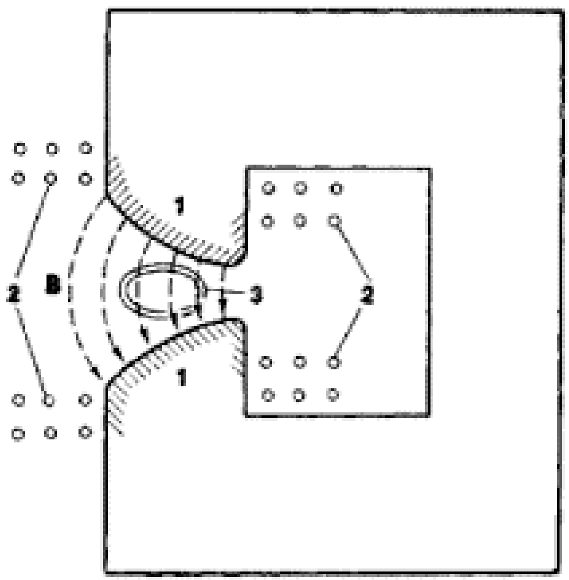
(a)

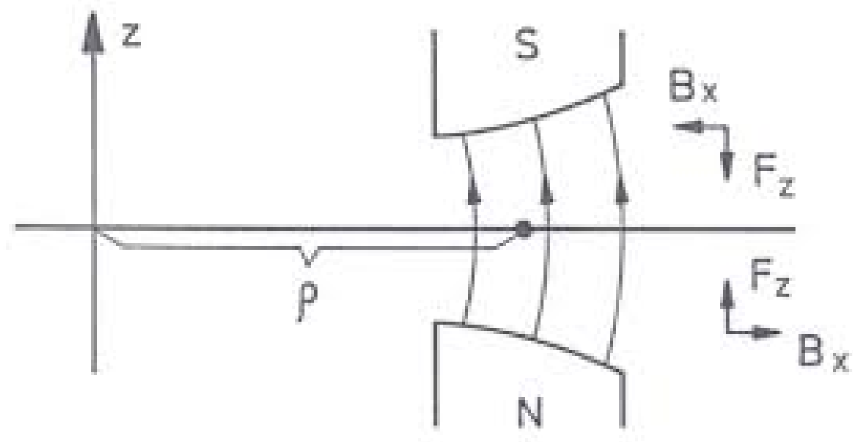
(b)

Two magnetic monopoles

- One can imagine the motion of an electric charge between two magnetic monopoles (of opposite polarity) would be integrable, but it is not.
  - Only the "adiabatic" integrals exist, when poles are far apart (compared to the Larmour radius)
  - This is the principle of the magnetic bottle; also, the principle of "weak focusing in accelerators".
- The non-integrability in this case is somewhat surprising because the motion in the field of two Coulomb centers is integrable.
  - This has been know since Euler and was Poincare's starting point for the 3-body problem quest.

Weak focusing

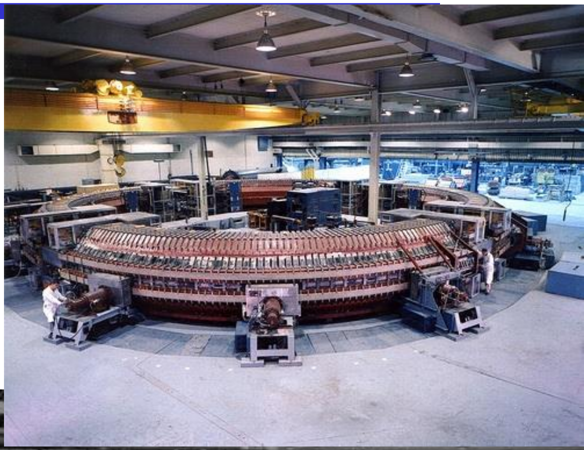




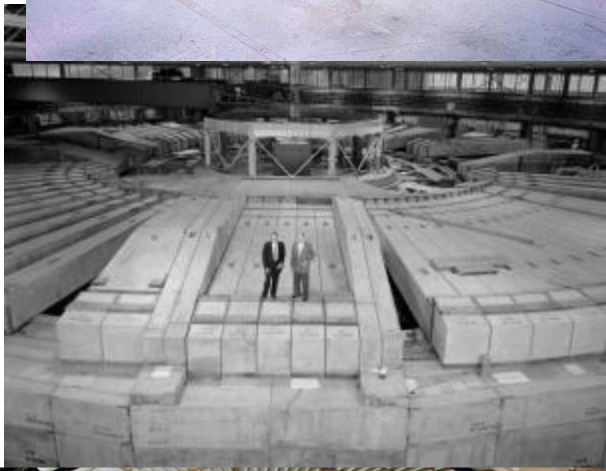
- The magnetic fields can be approximated by the fields of two magnetic monopoles of opposite polarity

The race for highest beam energy

- Cosmotron (BNL, 1953-66): 3.3 GeV
  - Produced "cosmic rays" in the lab
  - Diam: 22.5 m, 2,000 ton



- Bevatron (Berkeley, 1954): 6.3 GeV
  - Discovery of antiprotons and antineutrons: 1955
  - Magnet: 10,000 ton



- Synchrophasatron (Dubna,1957): 10 GeV
  - Diam: 60 m, 36,000 ton
  - Highest beam energy until 1959

Strong Focusing

PHYSICAL REVIEW VOLUME 88, NUMBER 5 DECEMBER 1, 1952

The Strong-Focusing Synchroton—A New High Energy Accelerator*

Ernest D. Courant, M. Stanley Livingston,† and Hartland S. Snyder
Brookhaven National Laboratory, Upton, New York
(Received August 21, 1952)

Strong focusing forces result from the alternation of large positive and negative n-values in successive sectors of the magnetic guide field in a synchrotron. This sequence of alternately converging and diverging magnetic lenses of equal strength is itself converging, and leads to significant reductions in oscillation amplitude, both for radial and axial displacements. The mechanism of phase-stable synchronous acceleration still applies, with a large reduction in the amplitude of the associated radial synchronous oscillations. To illustrate, a design is proposed for a 30-Bev proton accelerator with an orbit radius of 300 ft, and with a small magnet having an aperture of 1×2 inches. Tolerances on nearly all design parameters are less critical than for the equivalent uniform-n machine. A generalization of this focusing principle leads to small, efficient focusing magnets for ion and electron beams. Relations for the focal length of a double-focusing magnet are presented, from which the design parameters for such linear systems can be determined.

BETATRON OSCILLATIONS

RESTORING forces due to radially-decreasing magnetic fields lead to stable "betatron" and "synchrotron" oscillations in synchrotrons. The amplitudes of these oscillations are due to deviations from the equilibrium orbit caused by angular and energy spread in the injected beam, scattering by the residual gas, magnetic inhomogeneities, and frequency errors. The strength of the restoring forces is limited by the

* Work done under the auspices of the AEC.
† Massachusetts Institute of Technology, Cambridge, Massachusetts.

CERN PS

- In Nov 1959 a 28-GeV Proton Synchrotron started to operate at CERN
  - 3 times longer than the Synchrophasatron but its magnets (together) are 10 times smaller (by weight)

Particle motion in static magnetic fields

- For accelerators, there are no useful exactly integrable systems for axially symmetric magnetic fields in vacuum:

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{r} - \frac{eA_\theta(r,z)}{c}\right)^2$$

- Until 1959, all circular accelerators relied on approximate (adiabatic) integrability.
  - These are the so-called weakly-focusing accelerators
  - Required large magnets and vacuum chambers to confine particles;

Charge in a uniform magnetic field, *B*

- Let's use cylindrical coordinates: $r,\ \theta,\ z$

$$A_\theta = \frac{1}{2} Br; \quad B_z = B$$

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{r} - \frac{eA_\theta}{c}\right)^2$$

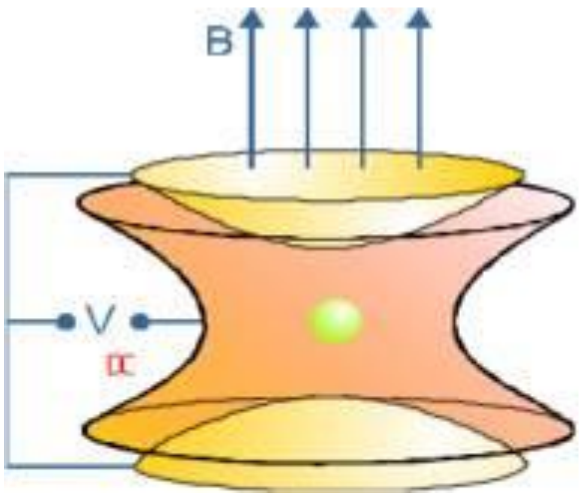
$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} - \frac{p_\theta \omega_c}{2} + \frac{m\omega_c^2 r^2}{8} \qquad \omega_c = \frac{eB}{mc}$$

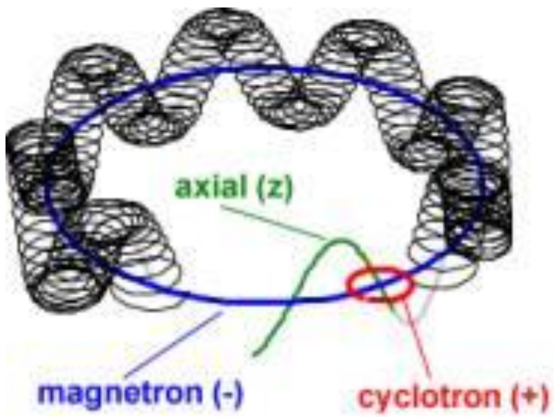
- This Hamiltonian is separable, however, the motion in *z* is unbounded (thus the action variable is undefined).

Penning trap

- An ideal Penning trap is a LINEAR and integrable system
  - It is a harmonic 3-d oscillator

$$H = \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3$$





Penning trap: equations of motion

$$A_\theta = \frac{1}{2}Br; \quad B_z = B \qquad\qquad \omega_c = \frac{eB}{mc}$$

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{1}{2m}\left(\frac{p_\theta}{r} - \frac{eA_\theta}{c}\right)^2 + \frac{eV_0}{4d^2}\left(2z^2 - r^2\right)$$

$$H = \frac{p_z^2 + p_r^2}{2m} + \frac{eV_0 z^2}{2d^2} + \frac{p_\theta^2}{2mr^2} + \frac{m\omega_c^2 r^2}{8} - \frac{eV_0 r^2}{4d^2} - \frac{p_\theta \omega_c}{2}$$

$p_\theta = const$ Integral of motion

- The system is bounded if $eV_0 > 0$ and $eV_0 < \dfrac{md^2\omega_c^2}{2}$

After some math, we obtain:

$$\omega_z = \sqrt{\frac{eV_0}{md^2}}; \quad \omega_r = \sqrt{\frac{\omega_c^2}{4} - \frac{eV_0}{2md^2}}$$

After variable separation

$$H = H_z + H_r$$

$$J_\theta = p_\theta = const$$

$$H_z = \frac{1}{2}\left(p_z^2 + \omega_z^2 z^2\right)$$

$$H_r = \frac{1}{2}\left(p_r^2 + \omega_r^2 r^2 + \frac{J_\theta^2}{r^2}\right)$$

$$z = \sqrt{\frac{2J_z}{\omega_z}} \sin\psi_z$$

$$r(J_r, \psi_r) = \sqrt{\frac{1}{\omega_r}}\left(2J_r + J_\theta + 2\sqrt{J_r^2 + J_r J_\theta}\sin\psi_r\right)^{1/2}$$

$$p_z = \sqrt{2J_z\omega_z}\cos\psi_z$$

$$p_r = \frac{2\sqrt{J_r^2 + J_r J_\theta}\cos\psi_r}{r(J_r, \psi_r)}$$

$$H = \omega_z J_z + 2\omega_r J_r + \omega_r J_\theta$$

Perturbation theory (introduction)

- Let's add a weak octupole nonlinearity to the Penning trap

$$V(r,z)=\frac{V_1}{d^4}\left(8z^4-24z^2r^2+3r^4\right)$$

$$H=\frac{p_z^2+p_r^2}{2m}+\frac{1}{2m}\left(\frac{p_\theta}{r}-\frac{eA_\theta}{c}\right)^2+\frac{eV_0}{4d^2}\left(2z^2-r^2\right)+\varepsilon eV(r,z)$$

- This is a non-integrable nonlinearity (…unlike a 1D non-linear oscillator).

- One can find approx. corrections by using a canonical perturbation method for $\varepsilon \ll 1$.

First order perturbation

Start with an integrable system $H = H_0(I)$

$$\dot{I} = -\frac{\partial}{\partial\theta} H_0(I) = 0$$

$$\dot{\theta} = \frac{\partial}{\partial I} H_0(I) = \omega(I)$$

$$H(I,\theta) = H_0(I) + \varepsilon H_1(I,\theta)$$

The goal is to find new J and φ such that $H(I,\theta) \to K(J)$

In the first order, we approximate $J = I$

$$H(J) \approx H_0(J) + \varepsilon \left\langle H_1(J,\theta) \right\rangle_\theta$$

How does one look for integrable systems for n>1?

- No general method of finding.

- One well-understood method (n=2): look for the second integral, quadratic in momenta
  - Turns out, all such potentials are separable in one of these 4 coordinate systems (cartesian, polar, elliptic, parabolic)
  - See Landau, Lifschitz "Mechanics" for reference
  - First comprehensive study by Gaston Darboux (1901)

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DES

SCIENCES

EXACTES ET NATURELLES

SUR UN PROBLÈME DE MÉCANIQUE

PAR

GASTON DARBOUX.

Dans son *Mémoire sur quelques unes des formes les plus simples que puissent présenter les intégrales des équations différentielles du mouvement d'un point matériel* publié en 1857 au *Journal de Mathématiques pures et appliquées* de Liouville, Joseph Bertrand aborde l'examen

Jean-Gaston Darboux
1842-1917

Darboux method

- We are looking for integrable potentials such that

$$H = \frac{p_x^2 + p_y^2}{2} + U(x, y)$$

and the second integral: $I = Ap_x^2 + Bp_x p_y + Cp_y^2 + D(x, y)$

$$[I, H] = 0$$

$$A = ay^2 + c^2,$$

$$B = -2axy,$$

$$C = ax^2,$$

Darboux equation (1901)

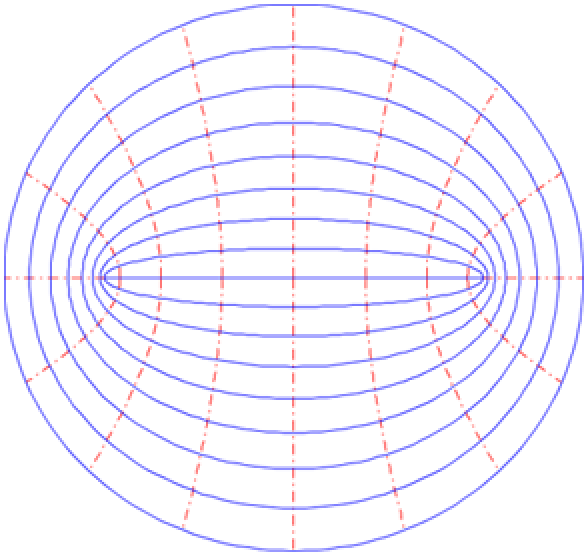
- Let $a \neq 0$ and $c \neq 0$, then we will take $a = 1$

$$xy\left(U_{xx} - U_{yy}\right) + \left(y^2 - x^2 + c^2\right)U_{xy} + 3yU_x - 3xU_y = 0$$

- General solution

$$U(x,y) = \frac{f(\xi) + g(\eta)}{\xi^2 - \eta^2}$$

$$\left.\begin{aligned} \xi &= \frac{\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}}{2c} \\ \eta &= \frac{\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}}{2c} \end{aligned}\right\} \text{Elliptical coordinates}$$



ξ : [1, ∞], η : [-1, 1], f and g arbitrary functions

The second integral

- The 2nd integral

$$I(x, y, p_x, p_y) = (xp_y - yp_x)^2 + c^2 p_x^2 + 2c^2 \frac{f(\xi)\eta^2 + g(\eta)\xi^2}{\xi^2 - \eta^2}$$

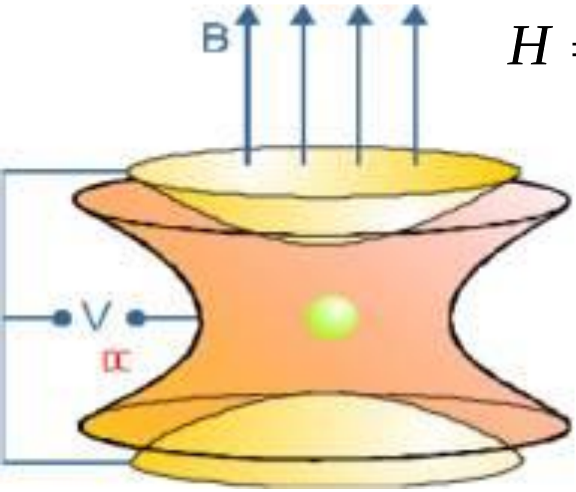
- Example:

$$U(x, y) = \frac{1}{2}\left(x^2 + y^2\right)$$

$$f_1(\xi) = \frac{c^2}{2}\xi^2\left(\xi^2 - 1\right) \qquad g_1(\eta) = \frac{c^2}{2}\eta^2\left(1 - \eta^2\right)$$

$$I(x, y, p_x, p_y) = (xp_y - yp_x)^2 + c^2 p_x^2 + c^2 x^2$$

What's the connection to traps?



$$H = \frac{1}{2m}\left(p_z^2 + p_r^2\right) + \frac{p_\theta^2}{2mr^2} - \frac{p_\theta \omega_c}{2m} + \frac{m\omega_c^2 r^2}{8} + \frac{eV_0}{4d^2}\left(2z^2 - r^2\right)$$

- Can one add an additional potential $U(r,z)$ such that the system becomes integrable and nonlinear?
  - The electrostatic potential must have

$$\Delta U(r,z) = 0$$

- Prolate spheroidal coordinates (u, v, ϕ)

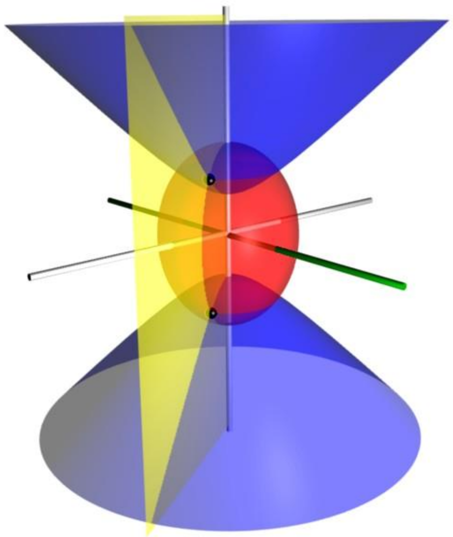
$$x = c\sqrt{\left(u^2-1\right)\left(1-v^2\right)}\cos\phi$$
$$y = c\sqrt{\left(u^2-1\right)\left(1-v^2\right)}\sin\phi$$
$$z = cuv$$

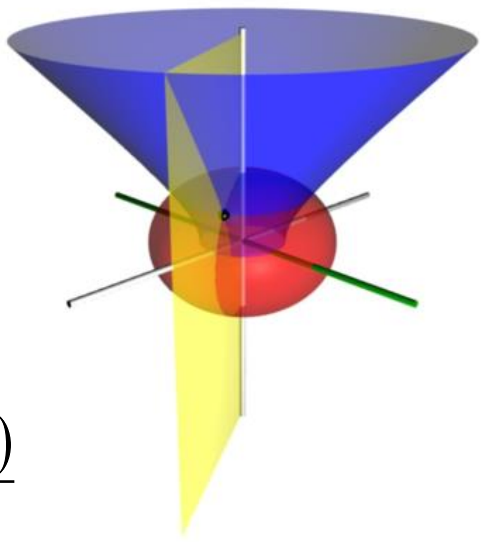
- Oblate spheroidal coordinates (ξ, η, ϕ)

$$x = a\sqrt{\left(1+\xi^2\right)\left(1-\eta^2\right)}\cos\phi$$
$$y = a\sqrt{\left(1+\xi^2\right)\left(1-\eta^2\right)}\sin\phi$$
$$z = a\xi\eta$$

- Separable potentials:

$$U(u,v) = \frac{g_1(u)+g_2(v)}{u^2-v^2}; \quad V(\xi,\eta) = \frac{f_1(\xi)+f_2(\eta)}{\xi^2+\eta^2}$$





Penning trap again

$$V_{eff}(r,z) = \frac{p_\phi^2}{2mr^2} - \frac{p_\phi \omega_c}{2m} + \frac{m\omega_c^2 r^2}{8} + \frac{eV_0}{4d^2}\left(2z^2 - r^2\right)$$

1 2 3

1 Is separable in both prolate and oblate spherical coordinates.

2 Is a constant

3 Is separable if $V_0 = \frac{mc^2}{e}\frac{\omega_c^2 d^2}{6c^2}$ or

$$V_{eff} = \frac{p_\phi^2}{2emr^2} - \frac{p_\phi \omega_c}{2em} + \frac{V_0}{2d^2}\left(z^2 + r^2\right)$$

Now add non-linear potentials

$$U(u,v)=\frac{g_1(u)+g_2(v)}{u^2-v^2};\quad V(\xi,\eta)=\frac{f_1(\xi)+f_2(\eta)}{\xi^2+\eta^2}$$

- In addition (the Laplace equation must be satisfied for electrostatic fields in vacuum):

$$\Delta U(r,z)=0 \qquad\qquad \Delta V(r,z)=0$$

- In prolate coordinates the solution has been know since Jacobi: the potential of Two Fixed Coulomb Centers (2FCC):
  - https://www.astro.auth.gr/~varvogli/two-fixed-centers.pdf

$$U(x,y,z)=\frac{\sigma_2}{\sqrt{(z-c)^2+r^2}}+\frac{\sigma_3}{\sqrt{(z+c)^2+r^2}}$$

2 Fixed Coulomb Centers

THE TWO FIXED CENTERS: AN EXCEPTIONAL INTEGRABLE SYSTEM

H. VARVOGLIS[1], CH. VOZIKIS[1,2] and K. WODNAR[3,4]

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(Received: 16 October 2003; revised: 8 May 2004; accepted 17 May 2004)

Abstract. It is usually believed that we know everything to be known for any separable Hamiltonian system, i.e. an integrable system in which we can separate the variables in some coordinate system (e.g. see Lichtenberg and Lieberman 1992, *Regular and Chaotic Dynamics*, Springer). However this is not always true, since through the separation the solutions may be found only up to quadratures, a form that might not be particularly useful. A good example is the two-fixed-centers problem. Although its integrability was discovered by Euler in the 18th century, the problem was far from being considered as completely understood. This apparent contradiction stems from the fact that the solutions of the equations of motion in the confocal ellipsoidal coordinates, in which the variables separate, are written in terms of elliptic integrals, so that their properties are not obvious at first sight. In this paper we classify the

https://www.astro.auth.gr/~varvogli/two-fixed-centers.pdf

Oblate spheroidal coordinates

- The solution is known since 1959: the Vinti potential

VOLUME 3, NUMBER 1 PHYSICAL REVIEW LETTERS JULY 1, 1959

NEW APPROACH IN THE THEORY OF SATELLITE ORBITS*

John P. Vinti
National Bureau of Standards, Washington, D. C.
(Received April 27, 1959)

$$V(\xi,\eta)=(\xi^2+\eta^2)^{-1}[f(\xi)+g(\eta)].$$

The most general solution of Laplace's equation compatible with such a form is

$$V=C+b_0\,\mathrm{Re}(\xi+i\eta)^{-1}+b_1\,\mathrm{Im}(\xi+i\eta)^{-1}$$
$$+b_2(\xi^2+\eta^2)^{-1}\left[2\xi\tan^{-1}\xi+\eta\ln\left(\frac{1+\eta}{1-\eta}\right)\right],$$

where the logarithmic term has a singularity everywhere on the z axis. We therefore place $b_2=0$ and we also place $C=0$ to make V vanish at infinity.

Applicable to traps:

$$V(x,y,z)=\frac{\sigma_1}{a}\frac{\left(4a^2-\left[\sqrt{(a+r)^2+z^2}-\sqrt{(a-r)^2+z^2}\right]^2\right)^{1/2}}{\sqrt{(a+r)^2+z^2}\sqrt{(a-r)^2+z^2}},$$

ORBIT DETERMINATION USING VINTI'S SOLUTION

DISSERTATION

Steven P. Wright, Major, USAF

AFIT-ENY-DS-16-S-067

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

ORBIT DETERMINATION USING VINTI'S SOLUTION

I. Introduction

1.1 Motivation

On 12 February 2009, an active US communications satellite, Iridium-33, and an inactive Russian satellite, Cosmos-2251, collided at an altitude of approximately 790 km. This marked the first known accidental collision between spacecraft payloads. The resulting debris was estimated to range between 1,000 and 2,000 objects greater than 10 cm and between 60,000 and 120,000 objects greater than 1 cm [1]. Later that same year, the USSTRATCOM Commander at the time, General Kevin Chilton, claimed that the Big Space Theory came to a close with this seminal event. In other words, no longer can the space community claim the probability of collision with other orbiting objects, due to the sheer size of space, is low enough to be ignored [2].

Due to orbital changes experienced by the scattering of particles upon impact, this significant debris cloud has been dispersed near globally and now congests this highly populated altitude regime.[1] According to the US Air Force, as of 1 March 2013, it continued to track 2,160 pieces from this collision alone [3]. This number is approximately 1,530 as of 26 May 2015 [4].

In the days prior to the collision, a close approach of these two spacecraft was predicted by SOCRATES.[2] An avoidance maneuver was not performed prior to the collision,[3] and, with perfect hindsight, it could be surmised this was completely

[1] In the week prior to the incident, it was reported that approximately 1,050 objects would come within 5 km of any one of 66 satellites in the Iridium constellation

[2] SOCRATES is a public service to the international satellite community provided by the Center for Space Standards & Innovation (CSSI) - www.celestrak.com/SOCRATES

[3] Only Iridium-33 had the capability for maneuver

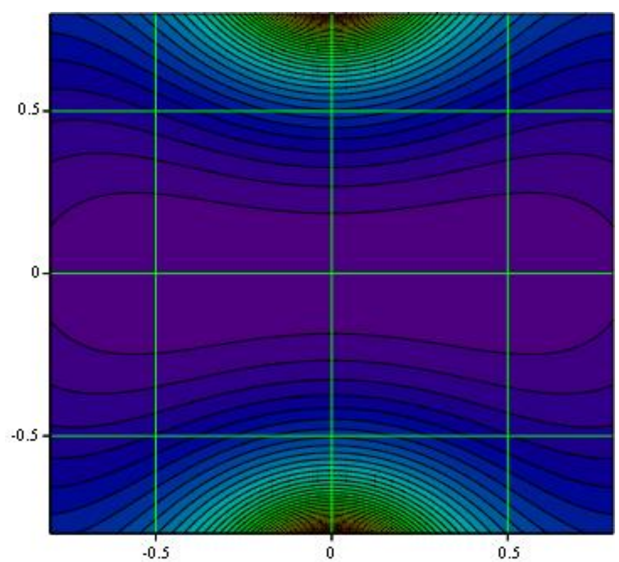
The non-linear trap recipe

1. Chose voltage (or magnetic field): $V_0 = \frac{mc^2}{e}\frac{\omega_c^2 d^2}{6c^2}$

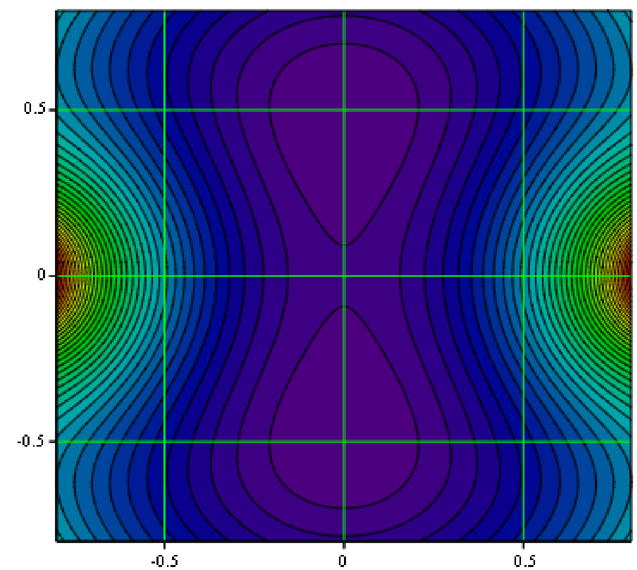
   For protons in 1-Tesla mag. field and *d* = 1 cm: $V_0 \approx 1.6\ \mathrm{kV}$

2. Add the non-linear potential, either 2FCC or Vinti

V_{eff}+ 2FCC



V_{eff}+ Vinti



Henon-Heiles paper (1964)

The Applicability of the Third Integral Of Motion: Some Numerical Experiments

Michel Hénon* and Carl Heiles

Princeton University Observatory, Princeton, New Jersey

(Received 7 August 1963)

The problem of the existence of a third isolating integral of motion in an axisymmetric potential is investigated by numerical experiments. It is found that the third integral exists for only a limited range of initial conditions.

$$U(r,\theta,z)=\frac{1}{2}\left(r^2+z^2\right)+r^2z-\frac{1}{3}z^3$$

Michel Henon (1988):

By a fortunate coincidence V. Arnold and J. Moser, working independently, had at the same time obtained their proofs of what was to become famous as the *KAM theorem*. In December 1962 I attended a gathering of astronomers at Yale. Moser was present and gave an illuminating presentation of the latest mathematical results and their consequences for the dynamics of nonintegrable systems. Suddenly everything fell into place: qualitatively at least, the mathematical theory completely explained the strange mixture of order and chaos found in our numerical results.

- First general paper on appearance of chaos in a Hamiltonian system.
- There exists two conserved quantities
  - Need 3 for integrability
- For energies $E > 0.125$ trajectories become chaotic
- Same nature as Poincare's "homoclinic tangle"

Henon-Heiles system

$$H=\frac{1}{2}\left(p_x^2+p_y^2+Ax^2+By\right)+Dx^2y-\frac{1}{3}Cy^3$$

Given the Henon-Heiles system with adjustable coefficients, are there any combinations that are integrable? Yes, the following 4 are known to be integrable:

$$D/C=0,\ \text{any } A,B$$

$$D/C=-1,\ A/B=1$$

$$D/C=-\frac{1}{6},\ \text{any } A,B$$

$$D/C=-\frac{1}{16},\ \mathrm{A}/B=\frac{1}{16}$$

Summary

- We have discussed a number of examples of integrable physical systems (linear and non-linear).

- Integrable systems form the "building blocks" about which perturbation theories can be developed.

- The non-integrable dynamic system actually constitute the majority of all dynamical systems. To find (or discover) integrable systems requires some luck.